Calculating Limits Using Limit Laws Lecture 2

It is not very practical nor rigorous to construct a table for each limit that we would like to calculate. The results below are fairly intuitive and they can be made rigorous with δ-ε arguments.

Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M$$

exist. Then

1. $\lim_{x\to a} \left(f(x)+g(x)\right) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

$= L+M$

2. $\lim_{x\to a} \left(f(x)-g(x)\right) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.

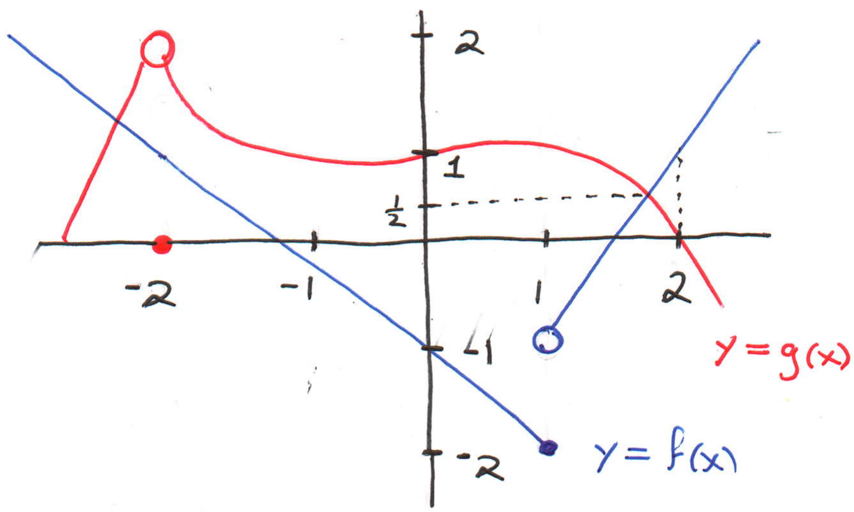
3. $\lim_{x\to a} c f(x) = c \lim_{x\to a} f(x) = cL$

4. $\lim_{x\to a} f(x) g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M$

5. $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$$

if $\lim_{x \to a} g(x) \neq 0$.

<u>Ex.</u> Use Limit Laws and the graphs of f and g to evaluate the following limits, if they exist.



(a) $\lim_{x \to -2} \left(f(x) + 5g(x) \right)$

(b) $\lim_{x \to 1} f(x)g(x)$

(c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$

Solution

(a) $\lim_{x \to -2} f(x) = 1$ and $\lim_{x \to -2} g(x) = 2$

so $\lim_{x \to -2} \left(f(x) + 5g(x)\right) = 1 + 5 \cdot 2 = 11.$

$$\boxed{\underbrace{f(x)}_{\downarrow \atop 1} + \underbrace{5g(x)}_{\downarrow \atop 5\cdot 2} \longrightarrow 1 + 5 \cdot 2}$$

(b) Note that $\lim_{x \to 1^+} f(x) = -1$ and $\lim_{x \to 1^-} f(x) = -2$

Hence $\lim_{x \to 1^+} f(x) g(x) = -1 \cdot 1$ while

$\lim_{x \to 1^-} f(x) g(x) = -2 \cdot 1$

The limit $\lim_{x \to 1} f(x) g(x)$ does not exist.

(c) $\lim_{x \to 2^-} g(x) = 0^+$ and $\lim_{x \to 2^+} g(x) = 0^-$

Since $\lim_{x \to 2} f(x) = 1$, we get

$$\lim_{x \to 2^-} \frac{f(x)}{g(x)} = \frac{1}{0^+} = +\infty$$

so the left-hand limit does not exist.

$$\lim_{x \to 2^+} \frac{f(x)}{g(x)} = \frac{1}{0^-} = -\infty$$

and this right-hand limit does not exist either.

It follows that $\lim_{x \to 2} \frac{f(x)}{g(x)} = \emptyset$.

The Limit Laws allow us to build an arsenal of functions, for which we right away know the limit.

<u>Ex.</u> $\lim_{x \to a} c = c$ and $\lim_{x \to a} x = a$

Therefore $\lim_{x \to a} x^2 = \lim_{x \to a} \overbrace{x}^{f(x)} \cdot \overbrace{x}^{g(x)} = \lim_{x \to a} x \cdot \lim_{x \to a} x$

$= a \cdot a = a^2$

$$\lim_{x \to a} x^3 = \lim_{x \to a} \overbrace{x^2}^{f(x)} \cdot \overbrace{x}^{g(x)} = a^2 \cdot a = a^3$$

and in general $\lim_{x \to a} x^n = a^n$ by repeated application of the limit product rule.

<u>Ex.</u> Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$ (b) $\lim_{x \to -2} \frac{x^3+2x^2-1}{5-3x}$

<u>Solution:</u>

(a) $\lim_{x \to 5} (2x^2-3x+4) = \lim_{x \to 5} (2x^2) - \lim_{x \to 5} (3x) + \lim_{x \to 5} 4$

(by Laws 2 and 1)

$= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + \lim_{x \to 5} 4$

(by 3)

$= 2 \cdot 5^2 - 3 \cdot 5 + 4$

(by 4)

$= 39.$

(b) $$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x \to -2} (x^3 + 2x^2 - 1)}{\lim_{x \to -2} (5 - 3x)}$$

(by Law 5)

$$= \frac{\lim_{x \to -2} x^3 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} 1}{\lim_{x \to -2} 5 - 3 \lim_{x \to -2} x}$$

(by 1, 2, and 3)

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}$$

Remark: Many novice Calculus students make the mistake and start believing that $\lim_{x \to a} f(x)$ and $f(a)$ are the same thing. This is largely due to the fact that the functions encountered in this course are largely polynomials or their close relatives - the rational functions.

Polynomials

Q. What are the simplest mathematical operations? What are the first mechanisms you learn as a pupil in elementary school?

A. You first encounter the operations of addition, subtraction, and multiplication. Subtraction may be regarded as the addition of negative numbers:

$$5-2 = 5+(-2).$$

The simplest function is therefore one which can be expressed as a sequence of additions and multiplications.

Ex. $f(x) = 5+2x-7(x+1)+2(x-3)^2$ uses only addition and multiplication. We can rearrange the terms according to how many times the input x is multiplied by itself:

$$f(x) = 5 + 2x - 7x - 7 + 2(x^2 - 6x + 9)$$

$$= \underbrace{(5 - 7 + 2\cdot 9)}_{x^0 \text{ (no } x)} + \underbrace{(2 - 7 - 2\cdot 6)x}_{x^1 \text{ (}\times x \text{ once)}} + \underbrace{2x^2}_{x^2 \text{ (}\times x \text{ twice)}}$$

$$= 16 - 17x + 2x^2$$

Def: A polynomial is a function $p(x)$ that can be written as

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

where $n \geq 0$ is a nonnegative integer and $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ (are real numbers)

Remark on notation: "$\in$" means "belongs to a set"

$\mathbb{N}$ = natural numbers = $\{1, 2, 3, \ldots\}$

$\mathbb{Z}$ = integers = $\{0, \pm 1, \pm 2, \pm 3, \ldots,\}$

The letter $\mathbb{Z}$ stands for Zahl, which is the German word for number.

$\mathbb{Q}$ = rational numbers = $\{ p/q : p, q \in \mathbb{Z}, q \neq 0 \}$

$\mathbb{Q}$ for quotient.

$\mathbb{R}$ = real numbers. It is a difficult matter to describe preceisely what numbers this category contains. This discussion would be lengthy and full of interesting paradoxes.

Most likely, every number you've encountered so far was a real number.

'$\subset$' means 'subset of'.

We have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

<u>Ex.</u> Identify which of the following functions are polynomials.

(a) $f(x) = 5$ (b) $g(x) = -3x^7 + 5x^2 + 10.$

(c) $h(x) = x^{-2} + 7x^3 + 3$

(d) $k(x) = \dfrac{x^3 + x^2}{x+1}$ (e) $v(x) = \dfrac{x^4 + x^2}{x^2+1}$

(f) $I(x) = 5x^{1/2} + 2x^3$

<u>Solution:</u>

(a) $f(x) = 5 + 0\cdot x + 0\cdot x^2 + \ldots +$

polynomial.

(b) $g(x) = 10 + 0\cdot x + 5x^2 + 0\cdot x^3 + \ldots + (-3)\cdot x^7$

polynomial.

(c) $h(x) = \dfrac{1}{x^2} + 7x^3 + 3.$ This involves division... but can we be sure that there is no way to describe this function without division?

Notice that $\frac{1}{x^2} + 7x^3 + 3 = \frac{7x^5 + 3x^2 + 1}{x^2}$

$$h(0) = \frac{7 \cdot 0^5 + 3 \cdot 0^2 + 1}{0^2} = \frac{1}{0} = \emptyset$$

but functions that operate using addition and multiplication are defined for all real numbers! Hence $h(x)$ cannot be expressed in the form

$a_0 + a_1 x + \ldots + a_n x$ and is therefore not a polynomial.

(d) $k(x) = \frac{x^2(x+1)}{(x+1)} = x^2$ if $x \neq -1$.

$k(-1)$ isn't defined so k is not a polynomial.

(e) $V(x) = \frac{x^2(x^2+1)}{(x^2+1)} = x^2$ for all x

Hence $V(x)$ is a polynomial.

(f) $I(x)$ isn't defined for $x < 0$ so $I(x)$ isn't a polynomial.

Thm: Let $p(x)$ be a polynomial, then $\lim_{x \to a} p(x) = p(a)$

Remark: This theorem says that applying limits on polynomials yields the same result as evaluating at the limit point.

Proof: Let $p(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n$

Then by repeatedly applying Limit Laws 1-4 (addition and multiplication) we get

$$\lim_{x \to a} p(x) = \lim_{x \to a} b_0 + \lim_{x \to a} b_1 x + \dots + \lim_{x \to a} b_n x^n$$

$$= b_0 + b_1 a + \dots + b_n a^n = p(a).$$

Polynomials are the simplest class of functions. Indeed, they remind us of whole numbers. Have you ever wondered why we say that our number system is base 10?

Consider, say $1806 = 6 + 0 \cdot \underbrace{10}_{x} + 8 \cdot \underbrace{10^2}_{x} + 1 \cdot \underbrace{10^3}_{x}$

so 1806 is like $6 + 8x^2 + x^3$

Q. Whats the next level of difficulty for functions?

A. The rational functions. Functions that use addition multiplication and division.

Ex. $R(x) = \frac{1}{x} + 2x + \frac{1}{x^2}$

$$= \frac{x}{x^2} + \frac{2x^3}{x^2} + \frac{1}{x^2} = \frac{1 + x + 2x^3}{x^2}$$

In general, you should convince yourself that whenever division is used, we can reduce to only one division operation.

Def: A rational map is a function $R(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

Ex. The following are rational functions

(a) $\frac{x^3 - 7x + 8}{5x^7 + 8x - 3}$ (b) Any polynomial (why?)

Thm: $\lim_{x \to a} R(x) = \lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$

unless $q(a) = 0$.

Proof: We already know that $\lim_{x \to a} p(x) = p(a)$

and $\lim_{x \to a} q(x) = q(a)$

Hence by the limit law for quotients (Law 5),

we have $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ unless $q(a) = 0$.

Ex. Find $\lim_{x \to 1} \frac{x^2-1}{x-1}$

Solution: $\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x \to 1} (x+1)$

$= 2.$

Ex. Evaluate $\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$

Solution:

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h}$$

$$= \lim_{h \to 0} \frac{6h + h^2}{h} = \lim_{h \to 0} \frac{h(6+h)}{h}$$

$$= \lim_{h \to 0} 6 + h = 6.$$

As it happens, $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$. This result is most easily established once we discuss derivatives of inverse functions.

<u>Ex.</u> Find $\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2}$

<u>Solution:</u>

1. $$\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} = \lim_{t \to 0} \frac{(\sqrt{t^2+9} - 3)(\sqrt{t^2+9} + 3)}{t^2(\sqrt{t^2+9} + 3)}$$

$$= \lim_{t \to 0} \frac{t^2 + 9 - 9}{t^2(\sqrt{t^2+9} + 3)} = \lim_{t \to 0} \frac{1}{\sqrt{t^2+9} + 3} =$$

$$= \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$$

2. $$\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} =$$

$$= \lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{(t^2+9) - 9}$$

$$= \lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{(\sqrt{t^2+9} - 3)(\sqrt{t^2+9} + 3)}$$

$$= \lim_{t \to 0} \frac{1}{\sqrt{t^2+9} + 3} = \frac{1}{6}.$$